Average Models, Z-Transforms and Matrix Algebra

An Average Model with Guaranteed Convergence

Introduction: Average models for pulse width modulators, PWM, have been developed by a number of authors, [1], [2], [3]. The common thread among them is to solve the differential equations for the switched current over a single cycle and replace that solution with an "average value". This approach eliminates the delay associated with integration and the resulting algebraic equation may not have a stable solution. The most common instability is caused when the solution involves either explicit or implicit quadratic equations; requiring selection of the "right" solution. This solution is usually "fixed up" by using soft limiters and in the general case doesn't guarantee a DC steady state solution. The steady state solution is needed in order to perform a small signal AC analysis in order to make a Bode plot. The Bode plot is used to design the compensation needed for a stable dynamic system.

Algebraic equations can be solved in general using matrix algebra. The matrix consists of a state matrix, multiplied by a state vector that produces an RHS (Right hand side) vector. The RHS and state vectors are column vectors with n states, and then the state matrix is an n x n square matrix. There are then n equations and n unknowns, so that if the equations are independent, there exists a solution. Besides independence, the equations must also be linear to give up a unique solution. Linear equations have only constant terms in the RHS. For a PWM circuit, the average output is the product of duty ratio times the input voltage, providing a non-linearity in that particular equation. That is, the RHS contains a product of 2 states. A non-linear equation solver such as SPICE3 solves the problem by decomposing the product into uv = Udv + Vdu, (U and V are the large signal average values and du and dv are the derivatives), then the components are moved back into the state matrix and the solution is iterated until U and V are stable. That is, the solution is linearized about the large signal operating point. Without the nonlinear term, the matrix solution can be found in one pass using Gauss elimination and backward substitution.

A similar problem exists in digital control systems. For digital controllers, the matrix needs to be linear so that it can be used over and over for each time step. The trick employed to linearize the equations is to insert a delay after the nonlinear term. The delayed product is then inserted as the initial condition for the next iteration. Then, knowing the initial condition, the steady state solution is easily found and the resulting AC analysis is guaranteed to converge. The initial condition can either be hand calculated or taken from the steady state of a transient analysis. The later method produces a generalized solution while the former method must be modified based on the detailed equations.

Now, for the Z-Transform model; the delay can actually approach 0, making the model exact. That suggests a new SPICE element, which we call STATESET to remind us of the similarity to the SPICE ".nodeset". STATESET cuts a network branch in the direction of signal flow. It behaves just like the ZDELAY model, but with the delay set to 0. The output is set to the initial condition for the DC operating point and the small signal gain is unity. For a TRAN simulation, the output is connected to the input. STATESET differs from ".nodeset" by enforcing the initial state variable for the DC operating point. That can't be done by ".nodeset" because SPICE nodes transmit current between components. For a controller, the signal is directional and the current can be neglected.

Yet Another PWM Model: To demonstrate this, a PWM model was developed from scratch for a buck regulator. This model can be generalized for a myriad of topology variations.

First, lets assume that the switched inductor will use a SPICE part as was done by Meares [2]. Then the sampled inductor current is measured using a SPICE voltage meter. Alternatively, the inductor state can be calculated by the model as done by BEN-YAAKOV [3]

Z-Transforms will be used to describe the system. Since Z-Transforms describe the system at sampling instants, it is necessary to calculate average values based on the sampled values. If the samples happen at the time when the current is at the average value, then no correction is needed!

First, it is necessary to configure the sampling switch in a manner that allows superposition of the averaged voltage. Figure 1 illustrates the switch position used here.
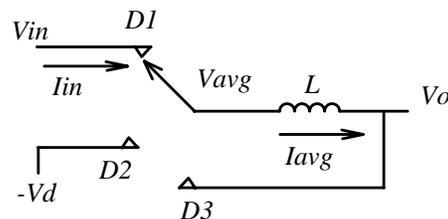
PWM Equations:

$$Vavg = D_1 Vin - D_2 Vd + D_3 Vo \qquad (1)$$

where: $D_n = T_n Fs$, and

$\quad T_n = $ Time at position n

$\quad Fs = $ Switching Frequency

$$Iin = D_1 Iavg$$

and all voltages are measured with respect to ground.

Figure 1, Switch configuration with low impedance driving point allows for superposition.

Figure 1 shows the average model equations. But further calculations are required to find the time at each switch position. Figure 2 shows the waveforms needed to make these calculations.
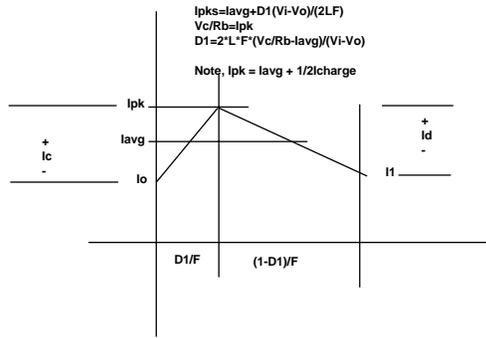
Figure 2, Basic waveform describing peak current control

From Figure 2,

$$I_1 = I_0 + I_C - I_D \qquad (2)$$

$$I_0 = I_1 z^{-1} \qquad (3)$$

$$I_{pk} = I_{avg} + \frac{I_C}{2} \qquad (4)$$

$$I_C = D_1 \frac{Vi - Vo}{LF_S} \cong D_{1P} \frac{Vi - Vo}{LF_S} \qquad (5)$$

$$I_{DD} = (1 - D_1) \frac{Vo + Vd}{LF_S} \cong (1 - D_{1P}) \frac{Vo + Vd}{LF_S} \qquad (6)$$

Where I_{DD} is a trial calculation for I_D assuming CCM

The peak current control law is:

$$\frac{V_C}{R_B} = I_{pk} \text{ , Where} \qquad (7)$$

V_C is the control input,

$\quad R_B$ is the current sense resistor.

Then solving for D_1 yields:

$$D_1 = 2LF_S \frac{\dfrac{V_C}{R_B} - I_{avg}}{Vi - Vo} \qquad (8)$$

$$D_{1P} = D_1 z^{-Kd} \qquad (9)$$

$$I_1 = I_0 + I_C + I_{DD} \qquad (10)$$

if $(I_1 < 0)$, DCM

$$I_D = I_{pk} = D_2 \frac{Vo + Vd}{LF_S} = I_{avg} + \frac{Ic}{2} \qquad (11)$$

$$D_2 = LF_S \frac{I_{avg} + \dfrac{Ic}{2}}{Vo + Vd} \qquad (12)$$

$$I_1 = 0$$

else CCM

$$D_2 = 1 - D_1 \cong 1 - D_{1P} \qquad (13)$$

and for both cases:

$$D_3 = 1 - (D_1 + D_2) \qquad (14)$$

Note: adding a constant to the denominator of (8) results in the Middlebrook equation, [4], where the constant is proportional to the ramp voltage. Adding ramp compensation can eliminate instability for D1 > .5 and improves phase margin of the implicit inner current control loop.

These equations can be implemented in IsSpice4 using B-elements and Intusoft's new ZDELAY model. Furthermore, it is necessary to sample Vc, using a "fast" ZDELAY sampler, or by inserting the new STATESET element, A5 in Figure 3. A zero order hold, X5, uses a ZDELAY element internally in order to simulate the PWM transport delay The initial duty ratio (D$_1$) and the initial inductor current, (Io) are required as model parameter inputs. With each of these values being a constant in the DC operating point calculation, it can be seen that the system of equations is now linear and can be solved directly without iteration. The simulation statistics found in the SPICE output file verify this.

This is essentially the same model that Intusoft uses in the FORWARD_ID model except for the delays, used to linearize the solution.

The model details and examples are available at: www.intusoft.com/newmodel.htm, including examples and test results comparing other popular average models.

Working with STATESET: STATESET is the new IsSpice4 code model used to guarantee DC convergence in non-linear circuit applications.

STATESET is a 2-terminal model having an input and output that is used to initialize its output node.

In the past, the simulator was allowed to iterate to a stable solution. But, many complex applications have arisen that have multiple stable operating points for which DC convergence fails.

STATESET forgoes the automatic DC operating point calculation. Instead it relies on user-based initialization. When used properly, the STATESET elements breakup nonlinear equations and Guarantees DC operating point convergence.

One notable equation is the "Middlebrook" equation [4] that describes a model for peak current control. Used in Figure 3, the B9 element implements this control law. This highly nonlinear equation involves the product of control voltage and average current divided by the switched inductor voltage. The small signal gain depends on the operating point of vc, the control input. When used with an external controller, it is necessary to use STATESET elements just before the vc, control term, and before the error amplifier as shown in Figure 4.

STATESET initial conditions can be estimated by the user and refined by running a transient solution to steady state. Then replace initial conditions with the measured steady state values. The control amplifier output in Figure 4 does not have to be exact; however, it must be within the linear operating region of the op-amp. While this is an extra step, it's well worth the effort because of the ease in getting AC analysis to run without convergence errors.

New PWM Models: Intusoft has developed a series of PWM average models that incorporate STATESET internally. They can be found in the parts browser, PWM Models-→AverageNew parts list. These models include example

drawings that can be accessed using the library manager to open the model and pressing "Test" to open the sample drawing.

Conclusion: Solution of non-linear algebraic equations may lead to multiple stable operating points. This may result in failure to converge on one of these operating points or convergence on the "wrong" answer. The dynamic or transient solution can be stable because there are delays within the control loop that prevent everything from being connected at the same time. These delays can be introduced artificially in the DC operating point calculation by inserting delay elements in the control loop and letting the delay time approach 0. This type of element is not native to SPICE; however, It can be implemented as an XSPICE code model. Intusoft's IsSpice4 is an XSPICE enabled simulator and models for a finite delay using an XSPICE model called ZDELAY plus a 0 time delay model called STATESET have been developed and tested. These models confirm the single step convergence characteristic resulting in a GUARANTEED operating point needed for a successful AC analysis.

References:
[1] R. D. MIDDLEBROOK and S. CUK, " A general Unified Approach to Modeling Switching Converter Power Stages ", IEEE PESC, 1976 Record, pp 18-34

[2]Meares, L.G. "New Simulation Techniques Using SPICE" Proceedings of the IEEE 1986 Applied Power Electronics Conference, pp 198-205

[3] Sam BEN-YAAKOV, "Average Simulation of PWM Converters by Direct Implementation of Behavioral Relationships", IEEE Applied Power Electronics Conference (APEC'93), pp 510-516

[4] Middlebrook, R.D. "Modeling Current-Programmed Buck and Boost Regulators", IEEE Transactions on Power Electronics, vol. PE-4, 1989 pp. 36-52
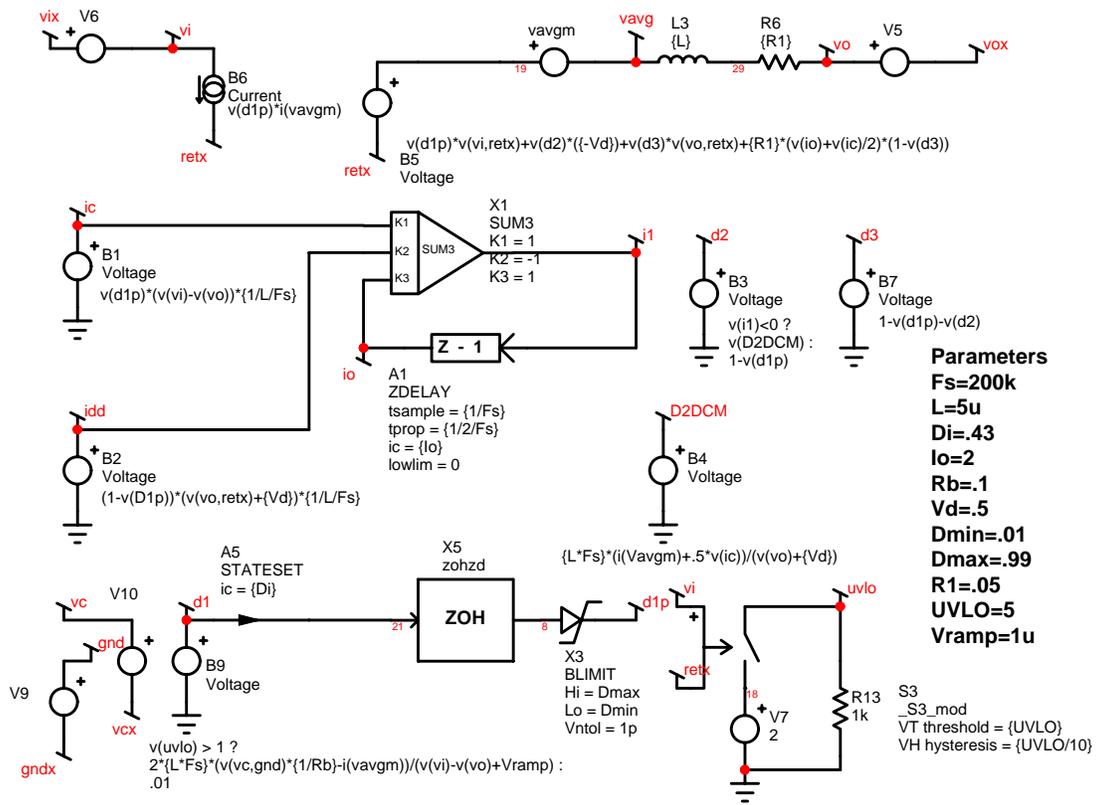
Figure 3, An IsSpice4 average PWM model for peak current control.
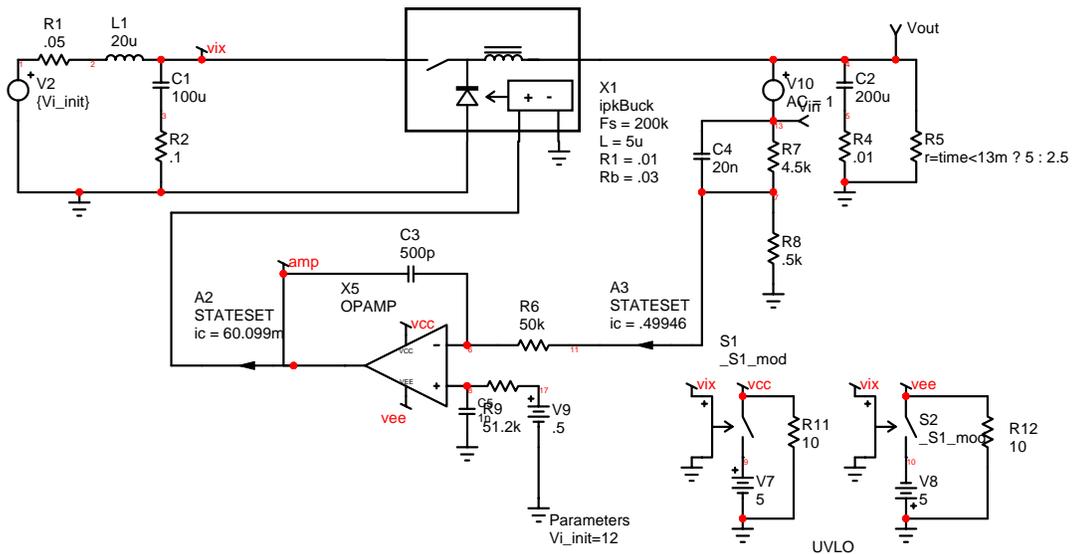

Figure 4, Example of peak current control PWM using external STATESET shown in A2 and A3.